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$$\begin{aligned}x^n+y^n &= (x+y-y)^n+y^n \\ &= (x+y)^n-n(x+y)^{n-1}y+\dots+n(x+y)y^{n-1},\end{aligned}$$

and since  $n$  is an odd prime, it follows that, if  $x^n+y^n$  is divisible by  $n$ ,  $x+y$  is also divisible by  $n$ , and hence  $x^n+y^n$  is divisible by  $n^2$ . From this follows directly the impossibility of the equation  $x^n+y^n=nz^n$  when  $z$  is prime to  $n$ .

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## AN INTERESTING CLASS OF MONOTONIC FUNCTIONS.\*

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By ARTHUR R. SCHWEITZER, Chicago, Illinois.

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1. In the *Rivista di Matematica*, Vol. 2, p. 43, Professor Peano has defined the following monotonic function. Representing the variable  $x$  of the interval  $(0, 1)$  by the series†

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{m^k} \qquad m \geq 2, \quad a_k \leq 1,$$

then he defines

$$t(x) = \sum_{k=1}^{+\infty} \left( \frac{a_k}{m^k} \right)^{n_k}$$

where the  $n_k$  are integers  $\geq 2$ . In this paper the function of Peano is generalized in such a manner that a very large class of monotonic functions is obtained which includes other types than the preceding; also an explicit evaluation of certain properties is given.

2. As is well known,‡ every point  $0 \leq x \leq 1$  can be represented by the series

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{\Pi_k}$$

where

$$\Pi_k = m_1 m_2 \dots m_k \qquad m_k \geq 2.$$

The point  $x$  has a unique representation when  $x=0$  or  $x=1$  or  $x=r/s$ , where  $r$  and  $s$  are relatively prime and  $s$  is not a divisor of the product  $\Pi_k$  for any  $k$ . In all other points  $x$  is irrational, and therefore has a unique represen-

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\*In this paper references will be made by notes to the following works: (1) Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, (2) Jordan's *Cours d'Analyse*, Vol. I, (3) Chrystal's *Algebra*. These will be briefly referred to as "Hobson," "Jordan," and "Chrystal," respectively.

†Chrystal I, p. 165. Representing  $X$  by the radix fraction  $a, a_2, a_3 \dots a_n \dots$  then we obtain  $t(x)$  by the suitable interpolation of zeros, e. g.,  $0 a_1, 0 a_2, 0 a_3 \dots 0 a_n \dots$

‡Chrystal, pp. 165, 166.

tation, or  $x$  is expressible by a finite number of terms and hence has a double representation. In the latter case  $x$  has the repetend zero ( $\phi$ ) or the repetend  $(m_k - 1)$  for an assigned  $k$  ( $m_k - 1$ ). Accordingly, we denote the points of the interval  $(0, 1)$  by  $x = x_f$  and  $x = x_\phi$ , respectively. If  $x = x_f$  and  $x = x_\phi$  we define  $x = x_\theta$ . Thus we have the three sets of representations,

$$X_f = [x_f], \quad X_\phi = [x_\phi], \quad X_\theta = [x_\theta].$$

Hence, in the well known notation of Cantor,\*

$$X_\theta = M(X_f, X_\phi),$$

and  $\beta(X_f, X_\phi)$  is the set of representations of points admitting expression by an infinite radix fraction only. We denote further

$$R(X_f, X_\phi) = M(X_f, X_\phi) - \beta(X_f, X_\phi),$$

so that  $R(X_f, X_\phi)$  is the set of representations of points expressible as a finite radix fraction.

3. We proceed to define the functions  $f(x)$ ,  $\phi(x)$ ,  $\theta(x)$ . If  $x = x_f$  we define

$$f(x) = \sum_{k=1}^{+\infty} r_k \frac{a_k l_k}{l'_k}$$

where

$$l'_k = m'_1, m'_2, \dots, m'_k, \quad m'_k \geq 2,$$

the numbers  $l_k$  are positive integers  $\geq 1$ , and

$$m_k \leq m'_k, \quad (m_k - 1)^{l_k} \leq m'_k - 1;$$

also the numbers  $r_k$  are positive. The function  $f(x)$  is then monotonic† if the numbers  $r_k$  form a non-increasing series. In an analogous manner we

$$f(x) = \sum_{k=1}^{+\infty} \frac{a_k}{(m_1 + 1)(m_2 + 1) \dots (m_k + 1)}.$$

define  $f(x)$  and  $\theta(x)$  corresponding to  $x = x_\phi$  and  $x = x_\theta$ . Thus  $f(x)$  and  $\phi(x)$  are single-valued, but  $\theta(x)$  is not. Denoting

$$F = [f(x)], \quad \Phi = [\phi(x)], \quad \Theta = [\theta(x)],$$

\*Hobson, §114; G. Cantor, *Mathematische Annalen*, 17, p. 355.

†Hobson, p. 245, §189. In verifying the argument of the text the reader is requested to select from one of the numerous functions defined above a particular example, e. g.,

we have

$$\Theta = M(F, \Phi),$$

and corresponding to  $\beta(X_f, X_\phi)$  and  $R(X_f, X_\phi)$  we have  $\beta(F, \Phi)$ ,  $R(F, \Phi)$ . As particular cases of the functions  $f(x)$ ,  $\phi(x)$ ,  $\theta(x)$  defined above, in addition to the preceding function of Peano, we may mention the functions of G. Cantor, *Mathematische Annalen*, 21, p. 590, and Liouville, *Journal de Mathématique*, 16, p. 133.

4. In this, and the following sections, we derive explicitly some of the properties of  $f(x)$ ,  $\phi(x)$ ,  $\theta(x)$  which we know in advance exist.\* The numbers  $f(x \pm 0)$ ,  $\phi(x \pm 0)$  are easily evaluated by a suitable selection of sequences. We have, in any point  $0 < x_0 < 1$ .

$$\begin{aligned} f(x_0 - 0) &= \phi(x_0 - 0), \\ f(x_0 + 0) &= \phi(x_0 + 0). \end{aligned}$$

If  $x_0$  is a point of  $R(X_f, X_\phi)$  then

$$\begin{aligned} f(x_0 - 0) &= \phi(x_0), \\ f(x_0 + 0) &= f(x_0). \end{aligned}$$

If  $x_0$  is a point of  $\mathcal{S}(X_f, X_\phi)$  then

$$\begin{aligned} f(x_0 - 0) &= f(x_0), \\ f(x_0 + 0) &= f(x_0). \end{aligned}$$

Thus the functions  $f(x)$  and  $\phi(x)$  are discontinuous in every point expressible as a finite radix fraction and continuous in every point expressible as an infinite radix fraction only.

5. The set  $\Theta$  of functional values  $\theta(x)$  is nowhere dense and perfect.† Hence the Cantor content‡ of  $\Theta$  is the difference between the length of the bounding interval and the sum of the point free intervals. For  $x$  on the set  $R(X_f, X_\phi)$  we have

$$f(x) - \phi(x) = r_\mu \cdot \frac{a_\mu l_\mu - (a_\mu - 1)l_\mu}{\Pi'_\mu} - \sum_{k=1}^{+\infty} r_{\mu+k} \frac{(m_k - 1)l_{\mu+k}}{\Pi'_{\mu+k}},$$

where

$$x = \sum_{k=1}^{\mu} \frac{a_k}{\Pi_k} \quad a_\mu \neq 0.$$

\*Jordan, §71.

†Hobson, §72.

‡Hobson, §85.

Since the number of these intervals is  $\Pi_{\mu-1}$ , where  $\Pi_0=1$ , we easily find the sum of the point free intervals to be

$$\begin{aligned} & \sum_{\mu=1}^{+\infty} \left[ \frac{\Pi_{\mu-1}}{\Pi'_{\mu-1}} \cdot \sum_{k=0}^{+\infty} r_{\mu+k} \cdot \frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu} \dots m'_{\mu+k}} - \frac{\Pi_{\mu}}{\Pi'_{\mu}} \cdot \sum_{k=1}^{+\infty} r_{\mu+k} \cdot \frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1} \dots m'_{\mu+k}} \right] \\ &= \sum_{k=1}^{+\infty} r_k \cdot \frac{(m_k-1)l_k}{\Pi'_k} - \lim_{\mu \rightarrow +\infty} \frac{\Pi_{\mu}}{\Pi'_{\mu}} \cdot \sum_{k=1}^{+\infty} r_{\mu+k} \cdot \frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1} \dots m'_{\mu+k}}. \end{aligned}$$

Hence the content of  $\Theta$  is

$$C = \lim_{\mu \rightarrow +\infty} \frac{\Pi_{\mu}}{\Pi'_{\mu}} \cdot \sum_{k=1}^{+\infty} r_{\mu+k} \cdot \frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1} \dots m'_{\mu+k}}.$$

Thus, if  $m'_k = m_k^{n_k}$ ,  $m_k \geq 2$ , the corresponding sets  $\Theta$  have content zero; a similar result is obtained in the case of the Liouville and Cantor functions mentioned above. Also  $\Theta$  has content zero if the series  $r_1 + r_2 + r_3 + \dots$ , is convergent. Again, if

$$\begin{aligned} r_{\mu+k} &= 1, \\ l_{\mu+k} &= 1, \\ m_{\mu+k} &= (\mu+k+1)^2 - 1, \\ m'_{\mu+k} &= (\mu+k+1)^2, \end{aligned}$$

the content of  $\Theta$  is the infinite product

$$\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \dots$$

which has a value different from zero.

6. When  $\Theta$  has content zero we can immediately rectify the curve of the function  $f(x)$  in the sense defined by Scheeffer,\* *Acta Mathematica*, Vol. 5, p. 59. We require the total discontinuity function of  $f(x)$ , say  $\Omega(x_0, x)$ . This function has for  $x_0 \leq x \leq 1$  the value given by the sum of the point free intervals of the subset of  $\Theta$  contained in the closed interval  $[f(x_0), \phi(x)]$ , plus the discontinuities of  $f(x)$  and  $\phi(x)$  in the points  $x_0, x$ . We have

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\*Jordan, p. 90 and following.

$$\Omega(x_0, x) = [\phi(x) - f(x_0)] + [f(x_0) - \phi(x_0)] + [f(x) - \phi(x)] = f(x) - \phi(x_0).$$

Hence

$$f(x) - \Omega(x_0, x) = \phi(x_0).$$

Corresponding to the closed interval  $(x_0, x_1)$  the length of the function  $f(x) - \Omega(x_0, x)$  is  $x_1 - x_0$  and hence the length of the curve of  $f(x)$  for this interval is

$$L = x_1 - x_0 + f(x_1) - \phi(x_0).$$

7. Finally, we determine the integral\*

$$\int_0^1 f(x) dx.$$

We divide the interval  $(0, 1)$  into  $\Pi_v$  equal parts. Then every partition point assumes the form

$$x_{.e_1 e_2 \dots e_v} = \sum_{k=1}^v \frac{e_k}{\Pi_k},$$

and conversely, every radix fraction so expressible is a partition point. Denoting the length of the interval whose first point is  $x_{.e_1 e_2 \dots e_v}$  by  $\delta_{.e_1 e_2 \dots e_v}$  we obtain,

$$\int_0^1 f(x) dx = \lim_{v \rightarrow +\infty} \sum_{(ek)} f(x_{.e_1 e_2 \dots e_v}) \delta_{.e_1 e_2 \dots e_v}$$

$$= \lim_{v \rightarrow +\infty} \sum_{(ek)} \sum_{k=1}^v r_k \cdot \frac{e_k^{l_k}}{\Pi'_k} \cdot \frac{1}{\Pi_v}.$$

Now the complete set of partition points (0 included, 1 excluded) may be divided into  $\Pi_v / m_k$  sets, each set being of the type

$$x_{.e_1 \dots e_{k-1} 0 e_{k+1} \dots e_v} \cdot x_{.e_1 \dots e_{k-1} 1 e_{k+1} \dots e_v} \cdot \dots \cdot x_{.e_1 \dots e_{k-1} m_k - 1 e_{k+1} \dots e_v}.$$

Hence in the complete set of partition points,  $e_k$  assumes the sequence of values

$$0, 1, \dots, m_k - 1$$

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\*Hobson, §225.

precisely  $II_v / m_k$  times. Hence

$$\begin{aligned} \int_0^1 f(x) dx &= \lim_{v \rightarrow +\infty} \sum_{k=1}^v \frac{r_k}{II_k} \cdot \frac{1}{II_v} \cdot \sum_{(ek)} e_k l_k \\ &= \lim_{v \rightarrow +\infty} \sum_{k=1}^v \frac{r_k}{II_k} \cdot \frac{1}{II_k} \cdot \frac{II_v}{m_k} \cdot \sum_{ek=1}^{m_k-1} e_k l_k \\ &= \sum_{k=1}^{+\infty} \frac{r_k}{m_k} \cdot \frac{1^{l_k} + 2^{l_k} + \dots + (m_k - 1)^{l_k}}{m'_1 m'_2 \dots m'_k}. \end{aligned}$$

For example, let

$$1^\circ. \quad l_k = 1, \quad r_k = 1, \quad m_k = 10, \quad m'_k = 10^3.$$

Then

$$\int_0^1 f(x) dx = \frac{1}{2}.$$

$$2^\circ. \quad l_k = 2, \quad r_k = 1, \quad m_k = 10, \quad m'_k = 10^3.$$

Then

$$\int_0^1 f(x) dx = \frac{1}{6} \frac{9}{8}.$$

Peano, *loc. cit.*, gives as the values of these integrals  $\frac{5}{111}$  and  $\frac{3^2}{111}$ , respectively.

8. In conclusion, I wish to thank Professor E. H. Moore for valuable suggestions in the preparation of this paper.

NOTE.—We regret to say that the copy of this article was lost in the mails on its transit to the author. In order to avoid longer delay of this number we undertook the proof reading of the article without the manuscript. Such errors as have escaped will be pointed out in a future issue. ED. F.